

## CELL COMPLEXES OBTAINED FROM SETS WITH RELATIONS

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**ABSTRACT.** For a positive integer  $r$ , an  $r$ -set we say in this paper is a set  $X$  with a subset of  $X^r$ . For  $r$ -sets  $X$  and  $Y$ , we construct the poset  $\text{Hom}(X, Y)$  called the Hom complex, and the simplicial set  $\text{Sing}(X, Y)$  called the singular complex. We show that their geometric realizations are homotopy equivalent. The Hom complex of  $r$ -sets is the generalization of the Hom complexes of graphs which have been applied to the graph coloring problem in combinatorics. On the other hand, singular complexes are more compatible with categorical construction of  $r$ -sets than Hom complexes. By using these complexes, the theory of beat points of posets, the  $\times$ -homotopy theory of graphs established by Dochtermann, and the strong homotopy theory of finite simplicial complexes established by Barmak and Minian can be unified to the case of  $r$ -sets.

## 1. INTRODUCTION

For a positive integer  $r$ , an  $r$ -set we say in this paper is a set  $X$  with a subset of  $X^r$ . We introduce the notions of the poset  $\text{Hom}(X, Y)$  called the Hom complex and the simplicial set  $\text{Sing}(X, Y)$  called the singular complex for  $r$ -sets  $X$  and  $Y$ .

The motivation of this article is the research of the Hom complexes of graphs. To explain this, we first state the history of topological combinatorics.

In [11], Lovász introduced the neighborhood complex  $N(G)$  of a graph  $G$  in the context of the graph coloring problem. He showed that the connectivity of the neighborhood complex gives a lower bound for the chromatic number, and determine the chromatic number of the Kneser graphs. The Hom complex of graphs was considered by Lovász after introducing the neighborhood complex, and was first mainly researched in [1].  $\text{Hom}(K_2, G)$  is homotopy equivalent to the neighborhood complex  $N(G)$  mentioned above. Lovász conjectured the connectivity of the Hom complex  $\text{Hom}(C_{2r+1}, G)$  where  $C_{2r+1}$  is the cycle graph with  $(2r+1)$ -vertices gives another lower bound for the chromatic number of  $G$ , and this conjecture was proved by Babson and Kozlov in [2], and the alternative proof is given by Schultz in [13]. After that, a number of researches on the Hom complexes of graphs have been found, for example, [6], [7], and [14]. For more concrete introduction and further researches of the Hom complexes of graphs, we refer the readers to [7] or [10].

Let us go back to the subject of  $r$ -sets. The Hom complex of  $r$ -sets is the poset which is a natural generalization of the Hom complex of graphs, and the minimal points of  $\text{Hom}(X, Y)$  can be identified with the map of  $r$ -sets, as is the case of graphs. The singular complex  $\text{Sing}(X, Y)$  is the simplicial set associated to  $r$ -sets  $X$  and  $Y$ , and its 0-simplex is identified with the map of  $r$ -sets. The first main result in this paper is that these two complexes are homotopy equivalent.

**Theorem 3.1** *Let  $X$  and  $Y$  be  $r$ -sets. Then there is a homotopy equivalence*

$$|\text{Sing}(X, Y)| \xrightarrow[\quad 1 \quad]{\simeq} |\text{Hom}(X, Y)|.$$

And this homotopy equivalence is natural with respect to  $X$  and  $Y$ . Moreover, for a map  $f : X \rightarrow Y$  of  $r$ -sets, this homotopy equivalence maps  $f \in |\mathrm{Sing}(X, Y)|$  to  $f \in |\mathrm{Hom}(X, Y)|$ .

As an application of the above theorem, we obtain another description of the homology groups of the Hom complexes which is similar to the singular homology groups of topological spaces.

The singular complex has merits from the viewpoint of the category theory rather than Hom complex. Indeed, for  $r$ -sets  $X, Y$ , and  $Z$ , Lemma 2.7 and Lemma 2.9 assert that there are natural equivalences

$$\mathrm{Sing}(X, Y \times Z) \cong \mathrm{Sing}(X, Y) \times \mathrm{Sing}(X, Z),$$

$$\mathrm{Sing}(X, Z^Y) \cong \mathrm{Sing}(X \times Y, Z).$$

On the other hand, the properties of Hom complexes corresponding to the above do not hold, but hold only up to homotopy equivalences.

In Section 4, we establish the strong homotopy theory of  $r$ -sets. As is mentioned above,  $r$ -sets are generalizations of graphs. For the sake of this generalizations to  $r$ -set, some theories which were considered in various contexts can be unified to the single theory of  $r$ -sets. The first is the  $\times$ -homotopy theory established by Dochtermann in [6], and the second is the strong homotopy theory of finite simplicial complexes by Barmak and Minian in [5].

Before closing this section, we fix some notations and introduce the terminology. Throughout this paper,  $r$  denotes a fixed positive integer. For a poset  $P$ , a subposet  $Q$  of  $P$  is said to be induced if for any  $x, y \in Q$ ,  $x \leq y$  in  $Q$  if and only if  $x \leq y$  in  $P$ . For a poset  $P$ , we write  $\Delta(P)$  for the order complex of  $P$ , and  $|P|$  for the geometric realization of  $\Delta(P)$ . For a poset map  $f : P \rightarrow Q$ , we say  $f$  is a homotopy equivalence if  $|f| : |P| \rightarrow |Q|$  is a homotopy equivalence. For poset maps  $f, g : P \rightarrow Q$ , we say  $f$  is homotopic to  $g$  if  $|f|$  is homotopic to  $|g|$ . Recall that for poset maps  $f, g : P \rightarrow Q$ , if  $f(x) \leq g(x)$  for any  $x \in P$  (in this case, we denote  $f \leq g$ ), then we have that  $f$  is homotopic to  $g$ .

## 2. HOM COMPLEXES AND SINGULAR COMPLEXES OF $r$ -SETS

In this section, we introduce the notions of  $r$ -sets, Hom complexes, and singular complexes of  $r$ -sets, and investigate their basic properties.

**2.1.  $r$ -sets.** Let  $X$  be a set. An  $r$ -relation of  $X$  is a subset of  $r$ -times direct product  $X^r$  of  $X$ . A pair  $(X, R)$  where  $X$  is a set and  $R$  is an  $r$ -relation of  $X$  is called an  $r$ -set. For an  $r$ -set  $(X, R)$ ,  $X$  is called the vertex set of  $(X, R)$ , and  $R$  is called the  $r$ -relation of  $(X, R)$ . We often abbreviate the  $r$ -relation, and say “ $X$  is an  $r$ -set”. In this notation, we write  $R(X)$  for the  $r$ -relation of the  $r$  set  $X$ , and  $V(X)$  for the vertex set of  $X$ . An  $r$ -set  $X$  is said to be finite if  $V(X)$  is finite.

Let  $X$  and  $Y$  be  $r$ -sets. A map of  $r$ -sets from  $X$  to  $Y$  is a set map  $f : V(X) \rightarrow V(Y)$  such that  $f^{\times r}(R(X)) \subset R(Y)$ , where  $f^{\times r}$  is the map  $X^r \rightarrow Y^r$  defined by  $(x_1, \dots, x_r) \mapsto (f(x_1), \dots, f(x_r))$ . We write  $\mathbf{Set}_r$  for the category of  $r$ -sets.

Let  $X$  and  $Y$  be  $r$ -sets. We define the  $r$ -set  $X \times Y$  by the direct product  $V(X) \times V(Y)$  with the  $r$ -relation

$$R(X \times Y) = \{((x_1, y_1), \dots, (x_r, y_r)) \mid (x_1, \dots, x_r) \in R(X), (y_1, \dots, y_r) \in R(Y)\}.$$

It is easy to see that  $X \times Y$  is the product object of  $X$  and  $Y$  in  $\mathbf{Set}_r$ .

*Example 2.1.* Recall that the 2-set  $(P, \leq)$  is called a pre-ordered set if  $\leq$  is transitive and reflexive. If  $\leq$  is anti-symmetric, then  $(P, \leq)$  is said to be partially ordered set.

*Example 2.2.* The 2-set  $G = (V(G), E(G))$  is called a graph if  $E(G)$  is symmetric.

**2.2. Hom complexes.** First we define the Hom complexes of  $r$ -sets. This is the straight generalization of the Hom complexes of graphs, investigated in [1], [2], [6], [7], [13], or [14], for example.

Let  $X$  and  $Y$  be  $r$ -sets. A *multi-map* from  $X$  to  $Y$  is a map  $\eta : X \rightarrow 2^Y \setminus \{\emptyset\}$  such that  $\eta(x_1) \times \cdots \times \eta(x_r) \subset R(Y)$  for any  $(x_1, \dots, x_r) \in R(X)$ . For multi-maps  $\eta$  and  $\eta'$  from  $X$  to  $Y$ , we write  $\eta \leq \eta'$  if  $\eta(x) \subset \eta'(x)$  for any  $x \in X$ . The *Hom complex from  $X$  to  $Y$*  is the poset of all multi-maps from  $X$  to  $Y$  with this ordering.

For a map  $f : X \rightarrow Y$  of  $r$ -sets, we can regard  $f$  as a multi-map  $x \mapsto \{f(x)\}$  for  $x \in X$  which is obviously a minimal element of  $\text{Hom}(X, Y)$ . And we often identified with a map of  $r$ -sets with a minimal point of the Hom complex.

A multi-map  $\eta$  from an  $r$ -set  $X$  to an  $r$ -set  $Y$  is said to be finite if  $\sum_{x \in X} (\#(\eta(x)) - 1) < \infty$ . We write  $\text{Hom}^f(X, Y)$  for the induced subposet of  $\text{Hom}(X, Y)$  whose vertex set is the set of all finite multi-maps from  $X$  to  $Y$ .

If  $X$  and  $Y$  are finite, then  $\text{Hom}(X, Y) = \text{Hom}^f(X, Y)$ . But in general,  $\text{Hom}(X, Y) = \text{Hom}^f(X, Y)$  does not hold. But we have that they are homotopy equivalent:

**Proposition 2.3.**  $|\text{Hom}^f(X, Y)|$  is a deformation retract of  $|\text{Hom}(X, Y)|$  for  $r$ -sets  $X$  and  $Y$ .

For graphs  $G$  and  $H$ , the fact that  $\text{Hom}(G, H)$  and  $\text{Hom}^f(G, H)$  has the same homotopy type seems to be known by experts, see [6] or [7]. But as far as I know, the proof of this fact is not yet written. So we write the precise proof of Proposition 2.3.

Proposition 2.3 is deduced from Lemma 2.4 and Lemma 2.6.

**Lemma 2.4.** Let  $P$  be an induced subposet of a poset  $Q$ . Suppose that for any  $y \in Q$ , we have  $P \cap (Q_{\leq y})$  is contractible. Then  $|P|$  is a deformation retract of  $|Q|$ .

Lemma 2.4 is the special version of the Quillen's fiber lemma A proved in [15]. But this is directly proved as follows. We should note that this proof is essentially the same one given in [4] for finite posets. But in [4], Barmak assumed that posets are finite, and his proof can not be just applied to the infinite case without a little modification given in the following.

*Proof.* By the Whitehead theorem, what we must prove is that  $\pi_i(|Q|, |P|, x)$  is trivial for any  $x \in P$ . Since taking the direct limit of cofibrations of topological spaces is commutative with taking homotopy groups, we can assume that  $Q \setminus P$  is finite.

Let  $x_1, \dots, x_n$  be the sequence of  $Q \setminus P$  satisfying the followings:

- $\{x_1, \dots, x_n\} = Q \setminus P$ .
- $x_i \neq x_j$  if  $i \neq j$ .
- If  $x_i \leq x_j$ , then  $i \leq j$ .

For  $i \in \{0, 1, \dots, n\}$ , we write  $Q_i$  for the induced subposet  $P \cup \{x_{i+1}, \dots, x_n\}$  of  $Q$ . Then  $Q_n = P$ ,  $Q_0 = Q$ , and  $Q_i \subset Q_{i-1}$  for  $i = 1, \dots, n$ . So what we must prove is that the inclusion  $Q_i \hookrightarrow Q_{i-1}$  is a homotopy equivalence for  $i \in \{1, \dots, n\}$ . Remark that

$$\Delta(Q_i) \cap \Delta(\text{st}_{\Delta(Q_{i-1})}(x_i)) = \Delta(P \cap Q_{\leq x_i}) * \Delta(Q_{> x_i}).$$

Since  $\Delta(P \cap Q_{\leq x_i})$  is contractible, we have that  $\Delta(Q_i) \cap \Delta(\text{st}_{\Delta(Q_{i-1})}(x_i))$  is contractible. Since  $\text{st}_{\Delta(Q_{i-1})}(x_i)$  is contractible, we have  $\Delta(Q_i) \cap \Delta(\text{st}_{\Delta(Q_{i-1})}(x_i))$  is a deformation retract of  $\Delta(\text{st}_{\Delta(Q_{i-1})}(x_i))$ . Therefore we have that  $\Delta(Q_i)$  is a deformation retract of  $\Delta(Q_{i-1}) = \Delta(Q_i) \cup \Delta(\text{st}_{\Delta(Q_{i-1})}(x_i))$ .  $\square$

*Remark 2.5.* By Lemma 2.4 and the same strategy of [4], we obtain an alternative proof of the Quillen's fiber lemma A for (infinite) posets. Suppose that  $f : P \rightarrow Q$  is an order preserving map such that  $f^{-1}(Q_{\leq y})$  is contractible for any  $y \in Q$ . What we want to prove is that  $f$  is a homotopy equivalence.

Let  $B(f)$  denote the poset whose base set is the direct sum  $P \sqcup Q$ , and the ordering of  $B(f)$  is defined as follows.

- (1)  $P$  and  $Q$  are induced subposets of  $B(f)$ .
- (2) For  $x \in P$  and  $y \in Q$ ,  $x \leq y$  in  $B(f)$  if and only if  $f(x) \leq y$  in  $Q$ .
- (3) There are no elements  $x \in P$  and  $y \in Q$  such that  $y < x$  in  $B(f)$ .

Let  $c : B(f) \rightarrow Q$  denote the map

$$c(x) = \begin{cases} f(x) & (x \in P) \\ x & (x \in Q). \end{cases}$$

Let  $i : P \hookrightarrow B(f)$  and  $j : Q \hookrightarrow B(f)$  denote the inclusion. Then we have  $jc \geq \text{id}_{B(f)}$  and  $cj = \text{id}_Q$ , and hence  $c$  induces a homotopy equivalence from  $B(f)$  to  $Q$ . On the other hand, by Lemma 2.4, we have that the inclusion  $i : P \hookrightarrow B(f)$  is a homotopy equivalence. Hence  $f = ci$  is a homotopy equivalence.

**Lemma 2.6.** *Let  $P$  be a non-empty poset. Suppose that for any  $x, y \in P$ , there is  $z \in P$  such that  $x \leq z$  and  $y \leq z$ . Then  $|P|$  is contractible.*

*Proof.* This is because any finite subposet of  $P$  is included in a contractible subposet.  $\square$

Let  $X, Y_1$ , and  $Y_2$  be  $r$ -sets, and  $f : Y_1 \rightarrow Y_2$  a map of  $r$ -sets. Then we define the poset map  $f_* : \text{Hom}(X, Y_1) \rightarrow \text{Hom}(X, Y_2)$  by  $\eta \mapsto (x \mapsto f(\eta(x)))$ .

Let  $X_1, X_2$  and  $Y$  be  $r$ -sets, and  $f : X_1 \rightarrow X_2$  a map of  $r$ -sets. Then we define the map  $f^* : \text{Hom}(X_2, Y) \rightarrow \text{Hom}(X_1, Y)$  by  $\eta \mapsto (x \mapsto \eta(f(x)))$ .

Remark that although  $\eta$  is a finite multi-map,  $f^*(\eta)$  is not necessarily finite. Hence for an  $r$ -set  $X$ ,  $\text{Hom}^f(-, X)$  is not a functor. On the other hand,  $\text{Hom}^f(X, -)$  is a functor for any  $r$ -set  $X$ . By Proposition 2.3, we have that the functor  $|\text{Hom}^f(X, -)|$  is naturally homotopy equivalent to  $|\text{Hom}(X, -)|$ .

For  $r$ -sets  $X, Y$ , and  $Z$ , the composition map

$$* : \text{Hom}(Y, Z) \times \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z), (\tau, \eta) \mapsto \tau * \eta$$

is defined by  $(\tau * \eta)(x) = \bigcup_{y \in \eta(x)} \tau(y)$ . Remark that  $*$  is a poset map. For a map  $f : X \rightarrow Y$  of  $r$ -sets and  $\tau \in \text{Hom}(Y, Z)$ , we have  $\tau * f = f^*(\tau)$ . On the other hand, for a map  $g : Y \rightarrow Z$  of  $r$ -sets and  $\eta \in \text{Hom}(X, Y)$ , we have  $g * \eta = g_*(\eta)$ .

**2.3. Singular complexes.** Next we construct a simplicial set  $\text{Sing}(X, Y)$  for  $r$ -sets  $X$  and  $Y$ , called the *singular complex from  $X$  to  $Y$* . Elementary facts and definitions of simplicial sets are found in [12].

We write  $\mathbb{N}$  for the set of all non-negative integers. For  $n \in \mathbb{N}$ , we write  $[n]$  for the poset  $\{0, 1, \dots, n\}$  with the usual ordering. For a non-negative integer  $n \in \mathbb{N}$ , the  $r$ -set  $\Sigma_n$  is defined by the set  $[n]$  with the  $r$ -relation  $[n]^r$ . Let  $T$  and  $X$  be  $r$ -sets.

The  $T$ -singular  $n$ -simplex of  $X$  is a map from  $\Sigma_n \times T$  to  $X$ . We write  $\text{Sing}(T, X)_n$  for the set of all  $T$ -singular  $n$ -simplices of  $X$ . Then  $\text{Sing}, (T, X)_\bullet$  becomes a simplicial set as a natural way. Namely, for an order preserving map  $f : [m] \rightarrow [n]$  the map  $f : \text{Sing}(T, X)_n \rightarrow \text{Sing}(T, X)_m$  is a map  $\sigma \mapsto ((i, a) \mapsto \sigma(f(i), a)$  where  $a \in T$  and  $i \in [m]$ . And the simplicial set  $\text{Sing}(X, Y)$  is called *the singular complex from  $X$  to  $Y$* . Since  $\Sigma_0 \times X \cong X$ , a 0-simplex of  $\text{Sing}(T, X)$  can be identified with a map of  $r$ -sets from  $T$  to  $X$  in the obvious way.

Let  $X, Y_1$ , and  $Y_2$  be  $r$ -sets, and  $f : Y_1 \rightarrow Y_2$  a map of  $r$ -sets. Then for an  $X$ -singular  $n$ -simplex  $\sigma : \Sigma_n \times X \rightarrow Y_1$ ,  $f \circ \sigma$  is an  $X$ -singular  $n$ -simplex of  $Y_2$ . And this defines a simplicial map  $f_* : \text{Sing}(X, Y_1) \rightarrow \text{Sing}(X, Y_2)$ .

Next let  $X_1, X_2$ , and  $Y$  be  $r$ -sets, and  $f : X_1 \rightarrow X_2$  a map of  $r$ -sets. Then for an  $X_2$ -singular  $n$ -simplex  $\sigma : \Sigma_n \times X_2 \rightarrow Y$  of  $Y$ ,  $\sigma \circ (\text{id}_{\Sigma_n} \times f) : \Sigma_n \times X_1 \rightarrow Y$  is an  $X_1$ -singular  $n$ -simplex of  $Y$ . And this defines a simplicial map  $f^* : \text{Sing}(X_2, Y) \rightarrow \text{Sing}(X_1, Y)$ .

As is the case of Hom complexes, we can define the composition map

$$* : \text{Sing}(Y, Z) \times \text{Sing}(X, Y) \rightarrow \text{Sing}(X, Z), (\tau, \sigma) \mapsto \tau * \sigma$$

as follows. Let  $\sigma : \Sigma_n \times X \rightarrow Y$  be an  $X$ -singular  $n$ -simplex of  $Y$  and  $\tau : \Sigma_n \times Y \rightarrow Z$  an  $Y$ -singular  $n$ -simplex of  $Z$ . Then the  $X$ -singular  $n$ -simplex  $(\tau * \sigma)$  of  $Z$  is defined by the composition

$$\Sigma_n \times X \xrightarrow{(p_1, \sigma)} \Sigma_n \times Y \xrightarrow{\tau} Z,$$

where  $p_1 : \Sigma_n \times X \rightarrow \Sigma_n$  is the first projection. And it can be shown that the composition map is compatible with  $f_*$  and  $f^*$  for a map  $f$  of  $r$ -sets.

Readers may have more complications for singular complexes than the case of Hom complexes. But there are some merits to consider singular complexes in the view of category theory. Indeed, the following lemma claimed that  $\text{Sing}(X, Y_1 \times Y_2)$  is naturally isomorphic to  $\text{Sing}(X, Y_1) \times \text{Sing}(X, Y_2)$ , although we can not say  $\text{Hom}(X, Y_1 \times Y_2) \cong \text{Hom}(X, Y_1) \times \text{Hom}(X, Y_2)$  in general.

**Lemma 2.7.** *Let  $X, Y$ , and  $Z$  be  $r$ -sets. Then the following hold.*

- (1) *There is a natural isomorphism  $\mathbf{Set}_r(X, Y \times Z) \cong \mathbf{Set}_r(X, Y) \times \mathbf{Set}_r(X, Z)$ .*
- (2) *There is a natural isomorphism  $\text{Sing}(X, Y \times Z) \cong \text{Sing}(X, Y) \times \text{Sing}(X, Z)$ .*
- (3) *There is a natural homotopy equivalence*

$$\text{Hom}(X, Y \times Z) \simeq \text{Hom}(X, Y) \times \text{Hom}(X, Z).$$

*Proof.* The proofs of (1) and (2) are straight forward, and are left to the reader.

Let us prove (3). Let  $p_1 : Y \times Z \rightarrow Y$  and  $p_2 : Y \times Z \rightarrow Z$  denote projections. Define the poset map

$$p : \text{Hom}(X, Y \times Z) \rightarrow \text{Hom}(X, Y) \times \text{Hom}(X, Z)$$

by  $p(\eta) = (p_{1*}(\eta), p_{2*}(\eta))$ . On the other hand, define the poset map

$$i : \text{Hom}(X, Y) \times \text{Hom}(X, Z) \rightarrow \text{Hom}(X, Y \times Z)$$

by  $i(\eta, \tau)(x) = \eta(x) \times \tau(x)$ . Then we have that  $pi = \text{id}$  and  $ip \geq \text{id}$ . Hence  $p$  is a homotopy equivalence.  $\square$

Next we define the morphism  $r$ -set  $Y^X$  for  $r$ -sets  $X$  and  $Y$ , which satisfy  $\mathbf{Set}_r(X \times Y, Z) \cong \mathbf{Set}_r(X, Z^Y)$ .

**Definition 2.8.** Let  $X$  and  $Y$  be  $r$ -sets. We define the  $r$ -set  $Y^X$  by

$$V(Y^X) = \{f : V(X) \rightarrow V(Y) \mid f \text{ is a set map.}\},$$

$$R(Y^X) = \{(f_1, \dots, f_r) \mid (f_1 \times \dots \times f_r)(R_X) \subset R_Y.\}.$$

**Lemma 2.9.** Let  $X, Y$  and  $Z$  be  $r$ -sets. Then the following hold :

- (1) There is a natural isomorphism  $\mathbf{Set}_r(X \times Y, Z) \cong \mathbf{Set}_r(X, Z^Y)$ .
- (2) There is a natural isomorphism  $\text{Sing}(X \times Y, Z) \cong \text{Sing}(X, Z^Y)$ .
- (3) There is a natural homotopy equivalence  $\text{Hom}(X \times Y, Z) \simeq \text{Hom}(X, Z^Y)$ .

*Proof.* Let  $\Phi : \mathbf{Set}_r(X \times Y, Z) \rightarrow \mathbf{Set}_r(X, Z^Y)$  be the map defined by

$$\Phi(f)(x)(y) = f(x, y) \quad (x \in X, y \in Y, f \in \mathbf{Set}_r(X \times Y, Z)),$$

and  $\Psi : \mathbf{Set}_r(X, Z^Y) \rightarrow \mathbf{Set}_r(X \times Y, Z)$  be the map defined by

$$\Psi(g)(x, y) = g(x)(y) \quad (x \in X, y \in Y, g \in \mathbf{Set}_r(X, Z^Y)).$$

Then we can prove that  $\Phi$  and  $\Psi$  are well-defined and the inverses of each other.

(2) is deduced from (1).

Let  $\Phi' : \text{Hom}(X \times Y, Z) \rightarrow \text{Hom}(X, Z^Y)$  be the map

$$\Phi'(\eta)(x) = \{f : V(Y) \rightarrow V(Z) \mid f(y) \in \eta(x, y) \text{ for } y \in V(Y)\},$$

and  $\Psi' : \text{Hom}(X, Z^Y) \rightarrow \text{Hom}(X \times Y, Z)$  be the map defined by

$$\Psi'(\eta)(x, y) = \{f(y) \mid f \in \eta(x)\}.$$

Then it is easy to show that  $\Phi'$  and  $\Psi'$  are well-defined poset maps. Let us show that  $\Phi' \circ \Psi' \geq \text{id}$  and  $\Psi' \circ \Phi' = \text{id}$ . Let  $\eta \in \text{Hom}(X, Z^Y)$ . Then we have

$$\begin{aligned} \Phi' \circ \Psi'(\eta)(x) &= \{f \mid f(y) \in \Psi'(\eta)(x, y) \text{ for } y \in V(Y)\} \\ &= \{f \mid f(y) \in \{g(y) \mid g \in \eta(x)\} \text{ for } y \in V(Y)\} \\ &\supset \eta(x). \end{aligned}$$

Hence we have that  $\Phi' \circ \Psi' \geq \text{id}$ . On the other hand, we have

$$\begin{aligned} \Psi' \circ \Phi'(\eta)(x, y) &= \{f(y) \mid f \in \Phi'(\eta)(x)\} \\ &= \{f(y) \mid f(y') \in \eta(x, y), \text{ for } y' \in V(Y)\} \\ &= \eta(x, y) \end{aligned}$$

Hence we have  $\Psi' \circ \Phi' = \text{id}$ . Therefore we have that  $\Phi'$  and  $\Psi'$  are homotopy equivalences. The naturalities of  $\Phi'$  and  $\Psi'$  can be easily proved.  $\square$

### 3. HOMOTOPY EQUIVALENCE

The purpose of this section is to show the following theorem.

**Theorem 3.1.** Let  $X$  and  $Y$  be  $r$ -sets. Then there is a homotopy equivalence

$$|\text{Sing}(X, Y)| \xrightarrow{\simeq} |\text{Hom}(X, Y)|.$$

And this homotopy equivalence is natural with respect to  $X$  and  $Y$ . Moreover, for a map  $f : X \rightarrow Y$  of  $r$ -sets, this homotopy equivalence maps  $f \in |\text{Sing}(X, Y)|$  to  $f \in |\text{Hom}(X, Y)|$ .

Recall that  $X \cong \Sigma_0 \times X \cong X \times \Sigma_0$  for any  $r$ -set  $X$ . Hence we have  $\text{Hom}^f(\Sigma_0, Y^X) \simeq \text{Hom}(\Sigma_0, Y^X) \simeq \text{Hom}(X, Y)$  and  $\text{Sing}(X, Y) \cong \text{Sing}(\Sigma_0, Y^X)$ . Let  $\text{Sing}(X)$  denote the singular complex  $\text{Sing}(\Sigma_0, X)$ . Hence to prove Theorem 3.1, it is sufficient to prove that there is a natural homotopy equivalence  $|\text{Sing}(X)| \rightarrow |\text{Hom}^f(\Sigma_0, X)|$ . First we investigate  $\text{Hom}^f(\Sigma_0, X)$ .

Let  $X$  be an  $r$ -set. A *clique* of  $X$  is a subset  $A$  of  $X$  such that  $A^r \subset R(X)$ . The  $r$ -set  $X$  is said to be a *clique* if  $X$  itself is a clique. The *clique complex* of  $X$  is the abstract simplicial complex

$$\text{Cl}(X) = \{A \subset X \mid A \text{ is a finite clique of } X\}.$$

Then  $\text{Hom}^f(\Sigma_0, X)$  is the face poset of  $\text{Cl}(X)$ , and hence we have that there is a natural homeomorphism  $|\text{Cl}(X)| \cong |\text{Hom}^f(\Sigma_0, X)|$ . So what we must do is to construct a homotopy equivalence  $f : |\text{Sing}(X)| \rightarrow |\text{Cl}(X)|$ .

Let  $K$  be a simplicial set. Recall that the geometric realization  $|K|$  of  $K$  is defined as follows. For each  $n$ -simplex  $\sigma$  of  $K$ , we write  $\Delta_\sigma$  for the canonical  $n$ -simplex, where the canonical  $n$ -simplex is the subspace

$$\Delta_n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum_{i=0}^n x_i = 1\}$$

of  $\mathbb{R}^{n+1}$ . Then  $|K|$  is obtained from the coproduct  $\coprod_{n \geq 0, \sigma \in K_n} \Delta_\sigma$  by identifying the following points.

- Suppose  $f : [m] \rightarrow [n]$  be an order-preserving map. Then we define the map

$$f_* : \Delta_{f^*\sigma} \rightarrow \Delta_\sigma, x_0 e_0 + \dots + x_m e_m \mapsto x_0 e_{f(0)} + \dots + x_m e_{f(m)},$$

where  $e_0, \dots, e_n$  are the standard basis of  $\mathbb{R}^{n+1}$ . Then in  $|K|$ , for  $x \in \Delta_{f^*\sigma}$ ,  $x$  and  $f_*(x)$  are identified.

Recall that the geometric realization  $|K|$  of a simplicial set  $K$  is a CW complex. Moreover, if  $L$  is a subsimplicial set of  $K$ , then  $|L|$  is a subcomplex of  $|K|$ .

The homotopy equivalence  $f_X : |\text{Sing}(X)| \rightarrow |\text{Cl}(X)|$  for an  $r$ -set  $X$  is defined as follows. Let  $\sigma$  be a map  $\Sigma_n \rightarrow X$  of  $r$ -sets. Define the continuous map  $f_\sigma : \Delta_\sigma \rightarrow |\text{Cl}(X)|$  by  $f_\sigma(\sum x_i e_i) = x_i \sigma(i)$ . Then for any poset map  $\varphi : [m] \rightarrow [n]$ , the diagram

$$\begin{array}{ccc} \Delta_{\varphi^*\sigma} & \longrightarrow & \Delta_\sigma \\ f_{\varphi^*\sigma} \downarrow & & \downarrow f_\sigma \\ |\text{Cl}(X)| & \xlongequal{\quad} & |\text{Cl}(X)| \end{array}$$

is commutative, and hence the maps  $f_\sigma$  defines a continuous map  $f_X : |\text{Sing}(X)| \rightarrow |\text{Cl}(X)|$ . To prove  $f_X$  is a homotopy equivalence, we need the following two lemmas.

**Lemma 3.2.** *Suppose  $X$  is a non-empty finite clique, namely,  $V(X)$  is a non-empty finite set and  $R(X) = V(X)^r$ . Then  $|\text{Sing}(X)|$  is contractible.*

*Proof.* First we remark that  $|\text{Sing}(\Sigma_0)|$  is the one point space. Hence if a map  $f : X \rightarrow Y$  of  $r$ -sets is a constant map, then  $f_* : |\text{Sing}(X)| \rightarrow |\text{Sing}(Y)|$  is also constant since  $f$  is factored as  $X \rightarrow \Sigma_0 \rightarrow Y$ .

Let  $X$  be a non-empty finite clique. It is easy to see that  $|\text{Sing}(X)|$  is connected. Indeed, for two 0-simplices  $a : \Sigma_0 \rightarrow X$  and  $b : \Sigma_0 \rightarrow X$ , we define the 1-simplex  $c : \Sigma_1 \rightarrow X$  by  $c(0) = a$  and  $c(1) = b$ . Then we have that  $d_1(c) = a$  and  $d_0(c) = b$ .

Define the map  $F : X \times \Sigma_1 \rightarrow X$  by  $F(x, 0) = x$  and  $F(x, 1) = x_0$ . Let  $i_k : \Sigma_0 \rightarrow \Sigma_1$  denote the map  $i_k(0) = k$  for  $k = 0, 1$ . Then the composition

$$X \cong X \times \Sigma_0 \xrightarrow{\text{id}_X \times i_k} X \times \Sigma_1 \xrightarrow{F} X$$

is equal to  $\text{id}_X$  if  $k = 0$  or is the constant map valued  $x_0$  if  $k = 1$ . Hence the composition

$$|\text{Sing}(X)| \xrightarrow{\text{id} \times |i_{k*}|} |\text{Sing}(X)| \times |\text{Sing}(\Sigma_1)| \xrightarrow{|F_*|} |\text{Sing}(X)|$$

is the identity if  $k = 0$ , and is the constant map if  $k = 1$ . Since  $\Sigma_1$  is a clique, we have that  $\text{Sing}(\Sigma_1)$  is connected. So we have that  $|\text{Sing}(X)|$  is contractible.  $\square$

The following lemma is a basic fact of topology, called the gluing lemma. See [8] for example.

**Lemma 3.3.** *Let  $X$  and  $Y$  be CW-complexes,  $I$  a set,  $\{A_i\}_{i \in I}$  an  $I$ -indexed family of subcomplexes of  $X$  such that  $\bigcup_{i \in I} A_i = X$ ,  $\{B_i\}_{i \in I}$  an  $I$ -indexed family of  $Y$  such that  $\bigcup_{i \in I} B_i = Y$ , and  $f$  a continuous map from  $X$  to  $Y$  such that  $f(A_i) \subset B_i$  for  $i \in I$ . Suppose that for any non-empty finite subset  $J$  of  $I$ , the map  $f|_{\bigcap_{j \in J} A_j} : \bigcap_{j \in J} A_j \rightarrow \bigcap_{j \in J} B_j$  is a homotopy equivalence. Then  $f : X \rightarrow Y$  is a homotopy equivalence.*

Let us show that the map  $f_X : |\text{Sing}(X)| \rightarrow |\text{Cl}(X)|$  is a homotopy equivalence. Remark that

$$\text{Sing}(X) = \bigcup_{A \in \text{Cl}(X)} \text{Sing}(A), \quad \text{Cl}(X) = \bigcup_{A \in \text{Cl}(X)} \text{Cl}(A).$$

and  $f_X$  maps  $|\text{Sing}(A)|$  to  $|\text{Cl}(A)|$ . By Lemma 3.2, we have that the restriction  $f_X|_{|\text{Sing}(A)|} : |\text{Sing}(A)| \rightarrow |\text{Cl}(A)|$  of  $f_X$  to  $|\text{Sing}(A)|$  is a homotopy equivalence for any finite clique  $A$  of  $X$ . Since for any  $A_1, \dots, A_k \in \text{Cl}(A)$ , we have that  $\text{Cl}(A_1) \cap \dots \cap \text{Cl}(A_k) = \text{Cl}(A_1 \cap \dots \cap A_k)$  and  $\text{Sing}(A_1) \cap \dots \cap \text{Sing}(A_k) = \text{Sing}(A_1 \cap \dots \cap A_k)$ , and hence they are both contractible, or both empty. By the gluing lemma, we have that  $f_X$  is a homotopy equivalence. This completes the proof of Theorem 3.1.

As an application of Theorem 3.1, we have another description of the homology of the Hom complex, which is similar to the singular homology of topological spaces. Let  $T$  and  $X$  be  $r$ -sets. Let  $C_n$  denote the free abelian group generated by  $\text{Sing}_n(T, X) = \{\sigma : \Sigma_n \times T \rightarrow X \mid \sigma \text{ is a map of } r\text{-sets}\}$ . and let  $\partial_i : \text{Sing}_n(T, X) \rightarrow \text{Sing}_{n-1}(T, X)$  denote the map  $\sigma \mapsto \sigma d_i$ . Define the abelian group homomorphism  $\partial : C_n \rightarrow C_{n-1}$  by  $\partial(\sigma) = \sum_{i=0}^n (-1)^i \partial_i(\sigma)$  for  $\sigma \in \text{Sing}_n(T, X)$ . Let  $\pi$  be an abelian group. It is known that the homology group  $H_*(C_n \otimes \pi)$  is isomorphic to the homology  $H_*(\text{Sing}(T, X), \pi)$ . Hence we have  $H_*(C_n \otimes \pi) \cong H_*(\text{Hom}(T, X))$ .

#### 4. STRONG HOMOTOPY OF $r$ -SETS

In this section, we establish the strong homotopy theory of  $r$ -sets. This is the generalization of the strong homotopy theory of finite simplicial complexes established by Barmak and Minian in [5], and the  $\times$ -homotopy theory of graphs established by Dochtermann in [6] to the case of  $r$ -sets.

First we begin the following definition.



**Definition 4.1.** Let  $X$  and  $Y$  be  $r$ -sets,  $f$  and  $g$  maps of  $r$ -sets from  $X$  to  $Y$ . We say that  $f$  is *strongly homotopic to  $g$*  if  $f$  and  $g$  are in the same connected component of  $\text{Hom}(X, Y)$ , and we write  $f \simeq_S g$ .

A map  $f : X \rightarrow Y$  of  $r$ -sets is called a strong homotopy equivalence if there is a map  $g : Y \rightarrow X$  of  $r$ -sets such that  $gf \simeq_S \text{id}_X$  and  $fg \simeq_S \text{id}_Y$ .

The goal of this section is to show that strong homotopy types of finite  $r$ -sets can be completely classified by their  $r$ -subsets called the *cores*. This fact is shown in several cases for example pre-ordered sets, finite simplicial complexes, and graphs. See [3], [5], and [6].

**Lemma 4.2.** Let  $X, Y$ , and  $Z$  be  $r$ -sets. The followings hold.

- (1) Let  $f$  and  $g$  be maps of  $r$ -sets from  $Y$  to  $Z$ . If  $f \simeq_S g$ , then  $f_* \simeq g_* : \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z)$ .
- (2) Let  $f$  and  $g$  be maps of  $r$ -sets from  $X$  to  $Y$ . If  $f \simeq_S g$ , then  $f^* \simeq g^* : \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z)$ .

*Proof.* We only give the proof of (1). The proof of (2) is similar. For  $\eta \in \text{Hom}(Y, Z)$ , let  $i_\eta$  denote the map  $\text{Hom}(X, Y) \rightarrow \text{Hom}(Y, Z) \times \text{Hom}(X, Y)$ ,  $\tau \mapsto (\eta, \tau)$ . Then we have  $* \circ i_f = f$  and  $* \circ i_g = g$ , where  $*$  is the composition map, see Section 2 for the definition. Hence we have that  $|*| \circ |i_f| = |f_*|$  and  $|*| \circ |i_g| = |g_*|$ . Since  $f$  and  $g$  are in the same connected component, there is a continuous map  $\varphi : [0, 1] \rightarrow |\text{Hom}(Y, Z)|$  such that  $\varphi(0) = f$  and  $\varphi(1) = g$ . Then the composition  $|*| \circ (\varphi \times \text{id}) : [0, 1] \times |\text{Hom}(X, Y)| \rightarrow |\text{Hom}(Y, Z)|$  gives the homotopy from  $|f_*|$  to  $|g_*|$ .  $\square$

**Corollary 4.3.** Let  $f : X \rightarrow Y$  be a map of  $r$ -sets. Then the followings are equivalent.

- (1)  $f$  is a strong homotopy equivalence.
- (2) For any  $r$ -set  $Z$ ,  $f_* : \text{Hom}(Z, X) \rightarrow \text{Hom}(Z, Y)$  is a homotopy equivalence.
- (3) For any  $r$ -set  $Z$ ,  $f^* : \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z)$  is a homotopy equivalence.

*Proof.* We only give the proof of (1)  $\Leftrightarrow$  (2). The proof of (1)  $\Leftrightarrow$  (3) is similar.

Suppose  $f$  is a strong homotopy equivalence. Then there is  $g : Y \rightarrow X$  such that  $gf \simeq_S \text{id}_X$  and  $fg \simeq_S \text{id}_Y$ . By Lemma 4.2, we have that  $g_* \circ f_* \simeq \text{id}$  and  $f_* \circ g_* \simeq \text{id}$ . Hence  $f_*$  is a homotopy equivalence.

Suppose  $f$  satisfies the condition (2). Then  $f_* : \text{Hom}(Y, X) \rightarrow \text{Hom}(Y, Y)$  is a homotopy equivalence, and hence there is  $g : Y \rightarrow X$  such that  $f_*(g) = f \circ g \simeq_S \text{id}_Y$ . Hence we have  $f_*(g \circ f) = f \circ (g \circ f) \simeq_S f = f_*(\text{id}_X)$ . Since  $\pi_0(f_*) : \pi_0(\text{Hom}(X, X)) \rightarrow \pi_0(\text{Hom}(X, Y))$  is injective, we have  $g \circ f$  and  $\text{id}_X$  are in the same connected component of  $\text{Hom}(X, X)$ , and hence we have  $g \circ f \simeq_S \text{id}_X$ . Therefore  $f$  is a strong homotopy equivalence.  $\square$

Let  $I_n$  be the  $r$ -sets defined as follows.

- $V(I_n) = [n]$ ,
- $R(I_n) = \bigcup_{1 \leq k \leq n} \{k-1, k\}^r$ .

Remark that  $I_1 \cong \Sigma_2$ . Hence the map  $I_n \times X \rightarrow Y$  is the sequence of 1-simplex of  $\text{Sing}(X, Y)$  such that the terminal point of a 1-simplex coincides the initial point of the next one.

**Proposition 4.4.** Let  $X$  and  $Y$  be  $r$ -sets,  $f$  and  $g$  maps of  $r$ -sets from  $X$  to  $Y$ . Then the following are equivalent.

- (1)  $f$  and  $g$  are in the same connected component of  $\text{Hom}(X, Y)$ , that is,  $f \simeq_S g$ .
- (2)  $f$  and  $g$  are in the same connected component of  $\text{Sing}(X, Y)$ .
- (3) There is  $n \in \mathbb{N}$  and a map  $I_n \times X \rightarrow Y$  of  $r$ -sets such that  $(0, x) \mapsto f(x)$  and  $(n, x) \mapsto g(x)$ .
- (4) There is  $n \in \mathbb{N}$  a map  $I_n \rightarrow Y^X$  of  $r$ -sets such that  $0 \mapsto f$  and  $n \mapsto g$ .

*Proof.* (1)  $\Leftrightarrow$  (2) is obtained from Theorem 3.1. (2)  $\Leftrightarrow$  (3) is obtained from the previous paragraph of this proposition. (3)  $\Leftrightarrow$  (4) is obtained from Lemma 2.6.  $\square$

**Definition 4.5.** Let  $X$  be an  $r$ -set, and let  $x \in X$ .  $x$  is said to be a *beat point* of  $X$  if there is  $y \in V(X) \setminus \{x\}$  such that  $(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \in R(X)$  for any  $i \in \{1, \dots, r\}$  and for any  $(x_1, \dots, x_n) \in R(X)$  such that  $x_i = x$ . If  $X$  has no beat points, we say that  $X$  is *minimal*.

*Example 4.6.* The notion of beat points of  $r$ -sets appears in various scenes in discrete mathematics or combinatorics as follows.

- (1) Let  $P$  be a pre-ordered set. In [4],  $x \in P$  is said to be upper beat point if  $P_{>x}$  has the minimum, and  $x$  is said to be lower beat point if  $P_{<x}$  has the maximum.  $x \in P$  is called “a beat point” if  $x$  is an upper or lower beat point. This notion of beat points coincides with Definition 4.5.
- (2) Let  $\Delta$  be a finite abstract simplicial complex, and  $r$  a positive integer greater than the dimension of  $\Delta$ . Then  $\Delta$  can be regarded as an  $r$ -set whose vertex set is the vertex set  $V(\Delta)$  of the simplicial complex  $\Delta$ , and the  $r$ -relation is  $\{(x_1, \dots, x_r) \mid \{x_1, \dots, x_r\} \in \Delta\}$ . For finite simplicial complexes  $\Delta_1$  and  $\Delta_2$  whose dimensions are smaller than  $r$ , the set map  $f : V(\Delta_1) \rightarrow V(\Delta_2)$  is a simplicial map if and only if  $f$  is a map of  $r$ -sets. In this notion, we have that  $v \in V(\Delta_1)$  is a beat point in Definition 4.5 if and only if  $v$  is the cone point in the sense of [5].
- (3) Let  $G$  be a graph. For  $v \in V(G)$ , we write  $N(v) = \{w \in V(G) \mid (v, w) \in E(G)\}$ . In [6],  $v$  is said to be *dismantlable* if there is  $w \in V(G) \setminus \{v\}$  such that  $N(v) \subset N(w)$ . The notion of dismantlable vertices coincides with beat points defined in Definition 4.5. Deleting a dismantlable vertex is called folding. The foldings of graphs do not change the homotopy types of Hom complexes was proved by Kozlov in [10], partially proved in [1].

Let  $X$  be an  $r$ -set. An  $r$ -subset  $Y$  of  $X$  is an  $r$ -set  $Y$  such that  $V(Y) \subset V(X)$  and  $R(Y) \subset R(X)$ . If  $R(Y) = V(Y)^r \cap R(X)$ , then  $Y$  is called an *induced  $r$ -subset* of  $X$ . This agrees the terminologies of “induced subposet” and “induced graph”.

**Lemma 4.7.** Let  $X$  be a minimal  $r$ -set and  $f : X \rightarrow X$  a map of  $r$ -set. If  $f \simeq_S \text{id}_X$ , then  $f = \text{id}_X$ .

*Proof.* Let  $\eta \in \text{Hom}(X, X)$  such that  $\text{id}_X < \eta$ . Then there is  $x \in X$  such that  $\{x\} \subsetneq \eta(x)$ . Let  $y \in \eta(x) \setminus \{x\}$ . Let  $i \in \{1, \dots, r\}$  and  $(x_1, \dots, x_r) \in R(X)$  such that  $x_i = x$ . Then we have that

$$(x_1, \dots, y, \dots, x_r) \in \eta(x_1) \times \dots \times \eta(y) \times \dots \times \eta(x_r) \subset R(X).$$

Hence  $x$  is a beat point, and  $X$  is not minimal.  $\square$

**Lemma 4.8.** Let  $X$  be a set and  $x$  a beat point of  $X$ . We write  $X \setminus \{x\}$  for the induced  $r$ -subset of  $X$  whose vertex set is  $V(X) \setminus \{x\}$ . Then the inclusion  $X \setminus \{x\} \hookrightarrow X$  is a strong homotopy equivalence.

*Proof.* Let  $i : X \setminus \{x\} \hookrightarrow X$  denote the inclusion. Since  $x$  is a beat point, there is  $y \in V(X) \setminus \{x\}$  such that for any  $i \in \{1, \dots, r\}$  and any  $(x_1, \dots, x_r) \in R(X)$  with  $x = x_i$ , we have that  $(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_r) \in R(Y)$ . Then the map  $f : V(X) \rightarrow V(X \setminus \{x\})$  defined by  $v \mapsto v$  for  $v \neq x$  and  $x \mapsto y$  is a map of  $r$ -sets. Then we have  $f \circ i = \text{id}$ . Let  $\eta$  denote the map  $V(X) \rightarrow 2^{V(Y)} \setminus \{\emptyset\}$  such that  $\eta(v) = \{v\}$  for  $v \neq x$  and  $\eta(x) = \{x, y\}$ . Then  $\eta$  is a multi-map such that  $i \circ f \leq \eta$  and  $\text{id}_X \leq \eta$ . Hence  $i \circ f \simeq_S \text{id}_X$ . Hence  $i$  is a strong homotopy equivalence.  $\square$

**Lemma 4.9.** *Let  $X$  and  $Y$  be minimal  $r$ -sets, and  $f : X \rightarrow Y$  a map of  $r$ -sets. If  $f$  is a strong homotopy equivalence, then  $f$  is an isomorphism.*

*Proof.* Let  $g : Y \rightarrow X$  be a strong homotopy inverse of  $f$ . Then we have  $gf \simeq_S \text{id}_X$  and  $fg \simeq_S \text{id}_Y$ . Hence by Lemma 4.7, we have  $gf = \text{id}_X$  and  $fg = \text{id}_Y$ . Hence  $f$  is an isomorphism.  $\square$

**Definition 4.10.** Let  $X$  be an  $r$ -set. The induced minimal  $r$ -subset  $Y$  of  $X$  is said to be a *core* of  $X$  if the inclusion  $Y \hookrightarrow X$  is a strong homotopy equivalence.

A finite  $r$ -set always has a core. But there is an infinite  $r$ -set which has no core. For example, the  $r$ -set  $L$  defined by

$$\begin{aligned} V(L) &= \mathbb{N} = \{0, 1, 2, \dots\}, \\ R(L) &= \{(x, y) \mid |x - y| = 1\} \end{aligned}$$

has no core. But it will be shown that a core of an  $r$ -set is unique up to isomorphisms if it exists.

**Theorem 4.11.** *Let  $X$  and  $Y$  be  $r$ -sets having cores. Then the followings are equivalent.*

- (1)  $X$  is strong homotopy equivalent to  $Y$ .
- (2) The core of  $X$  is isomorphic to the core of  $Y$ .

*Proof.* Since (2)  $\Rightarrow$  (1) is obvious, we only give the proof of (1)  $\Rightarrow$  (2). Let  $f : X \rightarrow Y$  be the strong homotopy equivalence. Let  $X'$  and  $Y'$  be cores of  $X$  and  $Y$ , respectively. Let  $i$  be the inclusion  $X' \hookrightarrow X$  and  $s : Y \rightarrow Y'$  be the strong homotopy inverse of the inclusion  $Y' \hookrightarrow Y$ . Then  $sfi$  is a strong homotopy equivalence, and hence is an isomorphism by Lemma 4.9.  $\square$

**Corollary 4.12.** *Let  $X$  be an  $r$ -set. Then the core of  $X$  is unique up to isomorphisms if it exists.*

**Acknowledgement.** The author would like to express his gratitude to Shouta Tounai for helpful suggestions especially on the proof of Proposition 2.3. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

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